Contact Geometry

a contact structure on a manifold M^{2n+1} is a special type of hyperplane field ξ

that is ξ is a subbundle of TM where fibers have dimension 1

we give the precise definition below

contact structures, along with symplectic structures, grew out of classical mechanics but it also has connections to PDE, Riemannian geometry, geometric optics, thermodynamics, ...

in the past 20 or so years it has also had many applications to 3-manifold topology

this class will cover the basic ideas in contact geometry but focus on dimension 3 and see how to classify contact structures on several manifolds as well as Legendrian knots (a special knot in a contact manifold)

I. Introduction

let M be an oriented 3-dimensional manifold

a plane field ξ on M is a subbundle of the tangent bundle of M such that

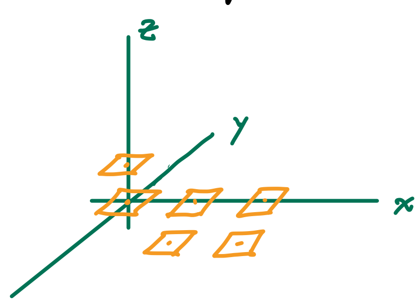
$$\xi_p = T_pM \cap \xi$$

is a 2-dimensional subspace of T_pM for all $p \in M$

Remark: compair to the more familiar vector field

examples:

1) in $\mathbb{R}^3$ let $\xi = \text{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$



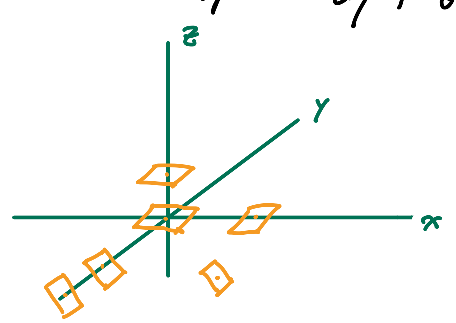
so at every $p \in \mathbb{R}^3$

ξ_p = tangent to a parallel copy of xy-plane through p

2) in $M = \Sigma^2 \times S^1$ let $\xi_p = T_x\Sigma \in T_p(\Sigma \times S^1)$

$p = (x, \theta)$

3) in $\mathbb{R}^3$ let $\xi = \text{span}\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\}$



4) let α be a 1-form on M

so $\alpha_p : T_pM \to \mathbb{R}$ is linear

we say α is non-singular if α_p is onto $\mathbb{R}$ for all p

assume α is non-singular

then $\xi_p = \ker \alpha_p \subset T_pM$ is a 2-plane for all p

$\therefore$ ξ is a plane field given by $\xi = \ker \alpha$

e.g. in example 1) $\xi = \ker(dz)$

2) $\xi = \ker(d\theta)$

3) $\xi = \ker(dz - y\,dx)$

exercise:

1) all plane fields can be locally written as $\ker\alpha$ for some α

(i.e. given ξ and $p \in M$, there is some open neighborhood U of p and a 1-form α on U s.t. $\xi_q = \ker \alpha_q$ for all $q \in U$)

2) ξ can be written as $\ker\alpha$, for some α on M

$\Longleftrightarrow$

$\exists$ a vector field v transverse to ξ at all points

this is called transversely orientable

$\Longleftrightarrow$

ξ is orientable

(for this exercise we need to recall M is orientable)

we will assume our contact structures are oriented

Frobenius Th$^{\underline{m}}$:

given an n-manifold M and

$F \subseteq TM$ a k-plane field

can find proof in many Diff. Top. books

Then

F is integrable $\Leftrightarrow$ F is closed under Lie brackets

an integral manifold of F through $p \in M$ is a one-to-one immersion $N \xrightarrow{f} M$ such that $df(T_x N) = F_{f(x)}$ and $p \in f(N)$

F is integrable if every point in M has an integral manifold through it

we say F is closed under Lie brackets if

$$v, w \in \underbrace{\Gamma(F)}_{\text{sections of } F} \Rightarrow \underbrace{[v, w]}_{\text{Lie bracket}} \in \Gamma(F)$$

(i.e. $\Gamma(F)$ is a Lie subalgebra of $\Gamma(TM)$)

suppose ξ is a plane field in TM^3, given as $\ker \alpha$ for some 1-form α

let $v, w \in \Gamma(\xi)$ so $v_p \in \xi_p = \ker \alpha_p$ so $\alpha(v) = 0$

similarly for w.

so $[v, w] \in \Gamma(\xi) \Leftrightarrow \alpha([v, w]) = 0$

Recall from Diff Top:

$$d\alpha(v, w) = v \cdot \alpha(w) - w \cdot \alpha(v) - \alpha([v, w])$$

so for $v, w \in \Gamma(\xi)$

$$[v, w] \in \Gamma(\xi) \Leftrightarrow d\alpha(v, w) = 0$$

coupled with Frobenius Th^{m} we see

ξ integrable $\Leftrightarrow d\alpha|_\xi = 0$

definition: ξ a plane field in TM given by $\ker\alpha$

ξ is a <u>foliation</u> if $d\alpha|_\xi = 0$

ξ is a <u>contact structure</u> if $d\alpha|_\xi$ is never zero

note: ξ contact $\Rightarrow$ a surface $\Sigma \subset M$ can never be tangent to ξ along an open set.

note: $d\alpha|_\xi \neq 0 \Leftrightarrow$ for any v, w spanning ξ $d\alpha(v,w) \neq 0$

let u be a vector field transverse to ξ

so $\alpha(u) \neq 0$

so we have
$$\alpha \wedge d\alpha(u,v,w) = \underbrace{\alpha(u)\cdot d\alpha(v,w)}_{\neq 0} + \underbrace{\text{terms with } v, w \text{ in } \alpha}_{\therefore 0}$$

thus we see

$$\boxed{\begin{gathered}\xi = \ker\alpha \text{ is contact} \\ \Leftrightarrow \\ \alpha\wedge d\alpha \text{ never zero}\end{gathered}}$$

note: a non-zero 3-form on M is a volume form so orients M

Remark: in higher dimensions M^{2n+1}

a contact structure is a hyperplane field ξ that is, locally, $\ker\alpha$ such that $\alpha\wedge(d\alpha)^n$ never zero

this condition is called "maximally non-integrable" and implies ξ cannot be tangent to a submanifold of dimension $k > n$ on an open set

Fact: contact manifolds must be odd dimensional since $\alpha\wedge(d\alpha)^n \neq 0$ implies that $d\alpha_p$ on ξ_p

is a non-degenerate 2-form (ie $\forall v \neq 0$ in ξ_p, $\exists$ $w \in \xi_p$ st. $d\alpha_p(v, w) \neq 0$)

ie. a linear symplectic structure

it is a fact that symplectic vector spaces must be even dimensional

exercise: Prove this

(you can find this in my notes on symplectic manifolds if you get stuck)

examples:

1) $\mathbb{R}^3$ with $\xi = \text{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\} = \ker \alpha$

where $\alpha = dz$

$d\alpha = d(dz) = 0$ so ξ is integrable

they are tangent to $\coprod_{z \in \mathbb{R}} \mathbb{R}^2 \times \{z\}$

a foliation of $\mathbb{R}^3$

2) $\Sigma \times S^1$ $\xi = \ker d\theta$

$d(d\theta) = 0$ so ξ is integrable

they are tangent to $\coprod_{\theta \in S^1} \Sigma \times \{\theta\}$

a foliation of $\Sigma \times S^1$

3) $\mathbb{R}^3$ $\xi = \ker \alpha$ $\alpha = dz - y\,dx$

$d\alpha = -dy \wedge dx = dx \wedge dy$

$\alpha \wedge d\alpha = dz \wedge dx \wedge dy$ a volume form on $\mathbb{R}^3$

so ξ is a contact structure on $\mathbb{R}^3$

note: if $\xi = \ker\alpha$ and $\xi = \ker\beta$ too then we must have $\alpha = f\beta$ for some non-zero function $f: M \to \mathbb{R}$

exercise: Prove this

thus
$$\alpha \wedge d\alpha = f\beta \wedge d(f\beta) = f\beta \wedge (df \wedge \beta + f d\beta)$$
$$= -f df \wedge \underbrace{\beta \wedge \beta}_{= 0} + f^2 \beta \wedge d\beta$$
$$= f^2 \beta \wedge d\beta$$

so we see <u>any</u> 1-form defining ξ will induce the same volume form on M

recall M is oriented so we can choose a volume form Ω on M inducing this orientation

since $\alpha \wedge d\alpha \neq 0$, there is some never zero function $g: M \to \mathbb{R}$ such that
$$\alpha \wedge d\alpha = g\,\Omega$$
(since $\Gamma(\Lambda^3 M) = C^\infty(M)$ if M oriented)

we write $\alpha \wedge d\alpha > 0$ if g is positive and $\alpha \wedge d\alpha < 0$ if g is negative

we call a contact structure
<u>positive</u> if $\alpha \wedge d\alpha > 0$
<u>negative</u> if $\alpha \wedge d\alpha < 0$

from now on "contact structure" means "positive contact structure"

exercise: 1) if ξ is a non-transversely oriented contact structure on M^3 (recall this means we can only define ξ as $\ker \alpha$ locally), then show ξ still defines an orientation on M

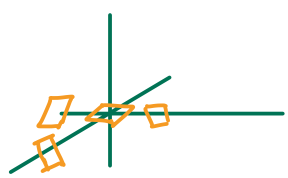
so non-orientable 3-manifolds do not have contact structures

2) is this true for contact structures on M^{2n+1}? Hint: consider parity of n

more examples: on $\mathbb{R}^3$ we have

1) $\alpha_1 = dz - y\,dx$ studied above

2) $\alpha_2 = dz + x\,dy - y\,dx = dz + r^2 d\theta$



twisting $\frac{\pi}{2}$ on all radial lines perpendicular to z-axis

are $\ker \alpha_1$ and $\ker \alpha_2$ the "same"

definition:

let (M_0, ξ_0) and (M_1, ξ_1) be contact manifolds

a contactomorphism is a diffeomorphism

$$f: M_0 \to M_1$$

such that $df(\xi_1) = \xi_2$

so a diffeomorphism taking one contact structure to the other

example:

$$f(x,y,z) = \left(\frac{x+y}{2}, \frac{y-x}{2}, z - \frac{xy}{2}\right)$$

is a diffeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ (check this)

and

$$f^*\alpha_2 = d\left(z - \frac{xy}{2}\right) - \frac{y-x}{2} d\left(\frac{y+x}{2}\right) + \frac{y+x}{2} d\left(\frac{y-x}{2}\right)$$

$$= dz - \tfrac{1}{2}(x\,dy + y\,dx) - \tfrac{1}{4}(y-x)(dy+dx) + \tfrac{1}{4}(x+y)(dy-dx)$$

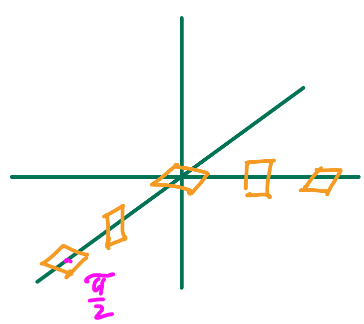
$$= dz - y\,dx + 0 + 0 + 0$$

$$= \alpha_1$$

$\therefore\ df(\xi_1) = \xi_2$ and $(\mathbb{R}^3, \xi_1)$ is contactomorphic to $(\mathbb{R}^3, \xi_2)$

3) $\alpha_3 = \cos r\, dz + r \sin r\, d\theta$

exercise: check $\ker \alpha_3$ is contact



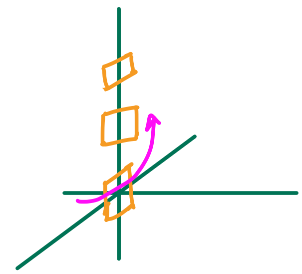
let $U_a = \{(r,\theta,z) \mid r < a\}$

exercise: 1) show $(U_a, \ker \alpha_3)$ is contactomorphic a subset of $(\mathbb{R}^3, \xi_2)$ for $a < \pi$

2) show $(U_a, \ker \alpha_3)$ is not contactomorphic to a subset of $(\mathbb{R}^3, \xi_2)$ for $a > \pi$

Hint: 2) is very hard, maybe come back to this later.

4) $\alpha_4 = \sin(2\pi z)\,dx + \cos(2\pi z)\,dy$



twists infinitely often on lines perp to xy-plane

exercise: show $\xi_4 = \ker \alpha_4$ is contactomorphic to ξ_1 and ξ_2

example: consider $S^3 \subset \mathbb{C}^2$ the unit sphere

recall for any linear space V, $T_pV \cong V$ canonically
so multiplication by i on $\mathbb{C}^2$ induces multiplication by i on all tangent spaces

let $\xi_p = T_pS^3 \cap i(T_pS^3) \quad \forall p \in S^3$

Claim: ξ is a contact structure on S^3

to see this recall $S^3 = f^{-1}(1)$ for

$$f : \mathbb{C}^2 \to \mathbb{R} : (x_1, y_1, x_2, y_2) \mapsto x_1^2 + y_1^2 + x_2^2 + y_2^2$$

and $T_{(x_1,y_1,x_2,y_2)} S^3 = \ker df_{(x_1,\dots,y_2)}$

$$= \ker 2(x_1\,dx_1 + y_1\,dy_1 + x_2\,dx_2 + y_2\,dy_2)$$

$$v \in i(TS^3) \iff iv \in TS^3 \iff df(iv) = 0$$

note: $i\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}, \quad i\frac{\partial}{\partial y_1} = -\frac{\partial}{\partial x_1}$

so $(dy_1 \circ i)\left(\frac{\partial}{\partial x_1}\right) = 1$ and $(dx_1 \circ i)\left(\frac{\partial}{\partial y_1}\right) = -1$

furthur computation gives

$$dx_1 \circ i = -dy_1 \quad dy_1 \circ i = dx_1 \quad \cdots$$

so $df \circ i = 2(-x_1 dy_1 + y_1 dx_1 - x_2 dy_2 + y_2 dx_2)$

$\therefore \alpha = (-x_1 dy_1 + y_1 dx_1 - x_2 dy_2 + y_2 dx_2)\big|_{TS^3}$ is a 1-form defining ξ

now $d\alpha = (-dx_1 \wedge dy_1 - dx_1 \wedge dy_1 - dx_2 \wedge dy_2 - dx_2 \wedge dy_2)$

$d\alpha \wedge d\alpha = 4\,(dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2)$

is the standard volumn form on $\mathbb{C}^2$

let $X = \frac{1}{2}\left(x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}\right)$

note that X is transverse to S^3 (since $df(X) > 0$)

$\iota_X d\alpha = \frac{2}{2}(-x_1 dy_1 + y_1 dx_1 - x_2 dy_2 + y_2 dx_2) = \alpha$

$\therefore \iota_X (d\alpha \wedge d\alpha) = 2\,\alpha \wedge d\alpha$

so if v_1, v_2, v_3 span T_pS^3, then

X, v_1, v_2, v_3 span $T_p\mathbb{C}^2$

$\therefore d\alpha \wedge d\alpha(X, v_1, v_2, v_3) \neq 0$

$\quad\quad \shortparallel$

$2\,\alpha \wedge d\alpha\,(v_1, v_2, v_3)$

and $\alpha \wedge d\alpha$ is a volumn form on S^3

$\therefore \xi = \ker \alpha$ contact as claimed!

<u>exercise</u>: $(S^3 - \{pt\}, \xi|_{S^3 - \{pt\}})$ is contactomorphic to $(\mathbb{R}^3, \xi_2)$ from above

Hint: see Geiges book if too hard

this is our first example of a contact structure on a closed manifold

later we will see that all closed oriented 3-mfds admit contact structures, for now

exercise:

1) construct a contact structure on T^3

Hint: $T^3 = \mathbb{R}^3/\mathbb{Z}^3$

consider ξ_4 on $\mathbb{R}^3$ above

2) construct a contact structure on the lens space

$L(p,q) = S^3/\mathbb{Z}_p$ where $S^3 \subset \mathbb{C}^2$ unit sphere and

$\mathbb{Z}_p$ acts by $(z_1, z_2) \mapsto (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi q i}{p}} z_2)$

we end our examples of contact structures with two important examples

example: Jet spaces

let M be a manifold

the 1-jet space of M is

$$J^1(M) = T^*M \oplus \mathbb{R}$$

given a function $f: M \to \mathbb{R}$ the 1-jet of f is

a section $\begin{array}{c} J^1(M) \\ \downarrow \\ M \end{array}$ given by

$j_1(f)(x) = (df_x, f(x))$

so $j_1(f)$ "remembers" f and its derivatives

there is a "canonical 1-form" λ on T^*M

it is uniquely determined by the property:

for all 1-forms β on M $\boxed{\beta^*\lambda = \beta}$

i.e. $\beta : M \to T^*M$

λ a 1-form on T^*M

$\beta^*\lambda$ is the pull-back of λ to M

and this pull-back is β

Exercise: show this uniquely defines a 1-form on T^*M

let's understand λ in local coordinates

let $(q_1, \dots, q_n)$ be local coords on $U \subset M$

these give coordinates on $T^*U = U \times \mathbb{R}^n$

$(q_1, \dots, q_n, p_1, \dots p_n)$

i.e. any 1-form on U can be written as $(q_1, \dots, q_n, \sum p_i \, dq_i)$

Claim: $\lambda = \sum p_i \, dq_i$

indeed a 1-form β in local coords is of the form

$$\beta(q_1, \dots, q_n) = \sum u_i(q_1, \dots q_n) \, dq_i$$
$$= (q_1, \dots, q_n, u_1, \dots u_n)$$

now $\beta^* \lambda = \sum u_i \, dq_i = \beta$

on $J^1(M) = T^*M \oplus \mathbb{R}$

let $\alpha = dz - \lambda$

where z is the coordinate on $\mathbb{R}$

note: $d\alpha = -\sum dp_i \wedge dq_i$

$(d\alpha)^n = n \, dp_1 \wedge dq_1 \wedge \dots \wedge dp_n \wedge dq_n$

is a volume form on T^*M

so $d\alpha \wedge (d\alpha)^n$ a volume form on $J^1(M)$

$\therefore \ \xi = \ker \alpha$ is a contact structure on $J^1(M)$

eg: $M = \mathbb{R}^1$ then $T^*\mathbb{R}^1 = \mathbb{R}^2$ $\quad \lambda = y\,dx$

and $J^1(\mathbb{R}^1) = \mathbb{R}^3$ $\quad \alpha = dz - y\,dx$

so jet spaces generalize our first example!

note: if $\phi : M \to N$ a diffeomorphism, then

$\phi^* : J^1(N) \to J^1(M) : (\beta, z) \mapsto (\phi^*\beta, z)$

induces a contactomorphism check this

Open Question: are M and N diffeomorphic

$\Longleftrightarrow$

$J^1(M)$ and $J^1(N)$ are contactomorphic note: given $f: M \to \mathbb{R}$

$$j_1(f) = (df, f)$$

$$(j_1(f))^*(dz - \lambda) = df - df = 0$$

so $\Gamma_{j_1(f)} = j_1(f)(M)$ is an n-dimensional submanifold of $J^1(M)$ and

$$T\Gamma_{j_1(f)} \subset \xi$$

def^n: if ξ^{2n} is a contact structure on M^{2n+1}

if $N^n \subset M^{2n+1}$ is a submanifold such that

$T_pN \subset \xi_p$ for all $p \in N$

then N is called a Legendrian submanifold

Legendrian submanifolds are very important geometric objects in contact geometry, we will study them a lot in dimension 3 (where they are knots)

so $\Gamma_{j_1(f)}$ is a Legendrian submanifold of $J^1(M)$

note: if $\sigma: M \to J^1(M)$ any section of $J^1(M)$

then $\boxed{\begin{array}{c} \sigma(M) \text{ is Legendrian} \\ \Leftrightarrow \\ \sigma = j^1(f) \text{ for some } f \end{array}}$

indeed if $\sigma = j^1(f)$ then done from above

if $\sigma(q_1, \dots q_n) = (q_1 \dots q_n, p_1(q), \dots, p_n(q), z(q))$

and $\sigma(M)$ Legendrian

then $\sigma^*(dz - \lambda) = 0$

$$dz_\sigma - \sum p_1(q_1 \ldots q_n)\, dq_n = 0$$

so $\dfrac{dz_\sigma}{dq_i} = p_i$ i.e. $\sigma = j_1(z_\sigma)$

check this!

Fun application: a 1^{st} order Partial Differential Equation

on $\mathbb{R}^n$ is a function

$$F : \mathbb{R}^{2n+1} \to \mathbb{R}$$

and $u : \mathbb{R}^n \to \mathbb{R}$ solves the equation

if $F(q_1, \ldots, q_n, \frac{\partial u}{\partial q_1}, \ldots, \frac{\partial u}{\partial q_n}, u) = 0$

eg $q_1 \dfrac{\partial u}{\partial q_1} \dfrac{\partial u}{\partial q_2} + u\, q_1 \dfrac{\partial u}{\partial q_2} = q_1^2 + 3q_1q_2 + 7$

associated F is

$$F(q_1, q_2, p_1, p_2, z) = q_1p_1p_2 + zq_1p_2 - q_1^2 - 3q_1q_2 - 7$$

note: F defines a function

$$F : J^1(\mathbb{R}^n) \to \mathbb{R}$$

s.t. u solves P.D.E. iff $F \circ j_1(u) = 0$

so solving a P.D.E. is equivalent to

finding a $\underbrace{\text{legendrian section } \sigma \text{ of } J^1(M)}_{\text{geometric}}$ s.t. $\underbrace{F \circ \sigma = 0}_{\text{algebraic}}$

Remark: Sophus Lie used this to study P.D.E.

He tried to find contactomorphisms of $J^1(\mathbb{R}^n)$ that simplified the algebra example:

let M be a manifold

fix a metric g on M

set UM = unit cotangent bundle

$= \{ \alpha \in T^*M \text{ s.t. } g(\alpha, \alpha) = 1 \}$

induced on T^*M by g

consider $v = \sum p_i \frac{\partial}{\partial p_i}$ using local coords from above

fiberwise radial field

let λ be cannonical form on T^*M from above

$\underbrace{d\lambda \wedge \dots \wedge d\lambda}_{n}$ = volume form on T^*M (from above)

$\iota_v(d\lambda) = \underbrace{\sum p_i \, dq_i}_{\lambda}$ (in local coords)

so $\iota_v(d\lambda \wedge \dots \wedge d\lambda) = n \, \lambda \wedge (d\lambda)^{n-1}$

this is a volume form on any space transverse to v

$\therefore \lambda \wedge (d\lambda)^{n-1}$ volume form on UM!

and $\xi = \ker \lambda$ a contact structure on UM

exercise: fibers of UM are Legendrian

exercise: given ω a 2-form on $\mathbb{R}^{2n+1}$ such that ω^n is never 0 (i.e. at any $p \in \mathbb{R}^{2n+1}$ $\exists v_1, \dots, v_{2n}$ s.t. $\omega^n(v_1, \dots, v_{2n}) \neq 0$)

then $\exists!$ vector $v \in \mathbb{R}^{2n+1}$, up to scale, s.t.

$\iota_v \omega = 0$

definition: given a contact form α on M^{2n+1}
then there is a unique vector field X_α
such that
$$\alpha(X_\alpha) = 1 \quad \text{and}$$
$$\iota_{X_\alpha} d\alpha = 0$$

X_α is called the Reeb vector field of α

this vector field is important to the study of $\xi = \ker\alpha$

note:
$$\mathcal{L}_{X_\alpha} \alpha = d\iota_{X_\alpha}\alpha + \iota_{X_\alpha} d\alpha = 0$$

Lie derivative

Cartan's magic formula

so the flow of X_α preserves α and hence ξ
we get a 1-parameter family of contactomorphisms of ξ from X_α !

examples:

1) $\mathbb{R}^3$ with $\xi = \ker(\underbrace{dz - y\,dx}_{\alpha})$
clearly $X_\alpha = \frac{\partial}{\partial z}$

2) for $\alpha = \lambda|_{UM}$ the Reeb vector field is
$$X_\alpha = \sum p_i \frac{\partial}{\partial q_i}$$

exercise: Check this

Fun application: connection to Riemannian geometry

let SM = unit tangent bundle of M

the geodesic flow on SM is

$$\Phi_t : SM \to SM$$
$$v \longmapsto \gamma_v'(t)$$

where $\gamma_v : \mathbb{R} \to M$ is a unit speed geodesic with

$\gamma_v(0) = \pi(v)$ $\qquad$ $\pi : SM \to M$ projection

$\gamma_v'(0) = v$

note: a flow line of Φ_t projects to a geodesic in M

can see all geodesics in M via Φ_t

$$g : SM \to UM : v \mapsto g(v, \cdot)$$

is a diffeomorphism $\quad$ check this!

Claim: g sends the geodesic flow to the Reeb flow

so we can study the geodesic flow, and hence much of Riemannian geometry, via contact geometry

to see the claim choose normal coordinates $q_1 \ldots q_n$ about a point $x \in M$

this means $x = (0, \ldots, 0)$ in these coords

$g_{ij}(0, \ldots, 0) = \delta_{ij}$

geodesics through x are $t \mapsto (ta_1, \ldots ta_n)$

so a geodesic flow line in SM above x is

$$t \mapsto (ta_1, \dots ta_n, a_1 \dots a_n)$$

and its tangent vector is

$$(a_1, \dots, a_n, 0, \dots, 0)$$

so the vector field generating the geodesic flow at $(0, \dots, 0, a_1, \dots a_n)$ is $\sum a_i \frac{\partial}{\partial \tilde{q}_i}$

we know the Reeb field at $(0, \dots, 0, a_1 \dots a_n)$ is

$$\sum a_i \frac{\partial}{\partial q_i}$$

the diffeo $SM \to UM$ induced by g is

$$(p_i, \tilde{q}_i) \mapsto (p_i, q_i)$$

so $\sum a_i \frac{\partial}{\partial \tilde{q}_i} \to \sum a_i \frac{\partial}{\partial q_i}$ at x

but this is true for <u>any</u> x